Summer term 2019 Research Institute for Discrete Mathematics
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Linear and Integer Optimization
Assignment Sheet 13
Sketches of Solutions

1. Show that $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not totally unimodular but $\{x \in \mathbb{R}^3 \mid Ax = b\}$ is integral for all integral vectors b.

   Solution idea: The 2×2-submatrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ on the upper left has determinant 2, so A cannot be TU. A solution $x = (x_1, x_2, x_3)$ to an integral vector $b = (b_1, b_2, b_3)$ is given by $x_1 = b_3$, $x_2 = b_2 + x_1$ and $x_3 = b_3 - x_1 - x_2$, so it is integral.

2. Let $A \in \{0,1\}^{m \times n}$ be a matrix where in each column the 1's are arranged consecutively, i.e. for each column $j \in \{1, \dots, n\}$ there are $i_1^j, i_2^j \in \{1, \dots, m\}$ s.t.:

$$a_{ij} = \begin{cases} 1, & i_1^j \le i \le i_2^j \\ 0, & \text{else} \end{cases}$$

   for $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, m\}$ (if $i_1^j > i_2^j$, the column consists of zeros only). Show that A is totally unimodular.

   Solution idea: By a theorem from the lecture, it is sufficient to show that for each set $R \subseteq \{1, \dots, m\}$ we can partition $R = R_1 \dot\cup R_2$ such that $\sum_{i \in R_1} a_i - \sum_{i \in R_2} a_i \in \{-1, 0, 1\}^n$ (where we denote the rows of A by a_i ($i = 1, \dots, m$)). This is always possible: for $R = \{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$ with $i_1 < i_2 < \dots < i_k$ set $R_1 = \{i_j \mid j \in \{1, \dots, k\}, j \text{ odd}\}$ and $R_2 = R \setminus R_1$. It is easy to check that R_1 and R_2 have the desired property.

3. Consider the following problem: We are given a directed graph G and nodes $s, t \in V(G)$ with $s \neq t$. Moreover, we are given integral mappings $l, u : E(G) \to \mathbb{Z}$ such that $l(e) \le u(e)$ for all $e \in E(G)$. The task is to find a mapping $f : E(G) \to \mathbb{R}$ with $l(e) \le f(e) \le u(e)$ for all edges $e \in E(G)$ and $\sum_{e \in \delta_G^-(v)} f(e) = \sum_{e \in \delta_G^+(v)} f(e)$ for all $v \in V(G) \setminus \{s, t\}$ such that $\sum_{e \in \delta_G^+(s)} f(e) - \sum_{e \in \delta_G^-(s)} f(e)$ is maximized. This problem generalizes the max-flow problem. Show that there is always an integral optimum solution and show that the value of a maximum solution equals

$$\min \left\{ \sum_{e \in \delta_G^+(X)} u(e) - \sum_{e \in \delta_G^-(X)} l(e) \mid X \subseteq V(G) \setminus \{t\}, s \in X \right\}.$$

   Solution idea: We can add an edge $e' := (t, s)$ with $l(e') = 0$ and $u(e') = \infty$. Call the resulting graph G'. Then, we ask for circulation maximizing the flow on (t, s). Thus, we can define a cost function $c : E(G') \to \mathbb{R}$ with $c((t, s)) = 1$ and $c(e) = 0$ for all other edges. Then, we can write the maximization problem in vector notation as $\max\{c^t x \mid I_m x \ge l, I_m x \le u, M_{G'} x = 0, x \in \mathbb{R}^{E(G')}\}$ where $M_{G'}$ is the incidence matrix of G'. Since $M_{G'}$ is TU and it remains TU after concatenating the identity matrix, the primal LP has an integral optimum solution. The dual LP is $\min\{u^t y_1 - l^t y_2 \mid z^t M_{G'} + y_1 - y_2 = c, z \in \mathbb{R}^{V(G)}, y_1, y_2 \in \mathbb{R}^{E(G)}\}$. It has as well an integral optimum solution $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{z}$, ans the objective function is the same as for the primal LP. We have $\tilde{y}_1(e') - \tilde{z}(s) + \tilde{z}(t) \geq 1$ Note that $\tilde{y}_1(e') = 0$ because $u(e') = \infty$, so we get $\tilde{z}(t) \geq 1 + \tilde{z}(s)$ By setting $U = \{v \in V(G) \mid \tilde{z}(v) < \tilde{z}(t)\}$, one gets a cut for which $\sum_{e \in \delta^+_G(U)} u(e) - \sum_{e \in \delta^-_G(U)}$ is as bis as the optimum solution value of the primal LP.

4. (a) Give an example of a polyhedron with $P_I \neq P^{(i)}$ for all $i \in \mathbb{N}$.
   (b) Show that for any $k \in \mathbb{N}$ there is a rational polyhedron such that $P_I \neq P^{(i)}$ for all $i \in \{1, \dots, k\}$.

Solution idea:

(a) Since $P^{(i)}$ is always closed, any polyhedron P for which $P_I \neq \emptyset$ is not closed works.
This is true e.g. for the polyhedron $P = \{(x, y) \ : \ x \geq 0, y \leq \sqrt{2}x\}$. We claim that $p := (1, \sqrt{2})$ is an accumulation point of P_I, but $p \notin P_I$.
First, assume $p \in P_I$ for the purpose of obtaining a contradiction. The inequality $y \leq \sqrt{2}x$ defines a face F of P that contains p. This means p maximizes $c := y - \sqrt{2}x$ over P.
By Carathéodory and the assumption $p \in P_I$, there is a convex combination $p = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2$ for $x_0, x_1, x_2 \in P \cap \mathbb{Z}^2$, with $\lambda_j \geq 0$ and $\lambda_0 + \lambda_1 + \lambda_2 = 1$. We have

$$c^T p = \lambda_0 c^T x_0 + \lambda_1 c^T x_1 + \lambda_2 c^T x_2,$$

and each $c^T x_i \leq c^T p$ by the observation above. Hence $c^T x_i = c^T p = 0$, i.e. the x_i all lie on the line through the origin orthogonal to c. The only integer point on this line is 0 (as in the previous part of the exercise), hence $x_0 = x_1 = x_2 = 0$ and $p = 0$. This is a contradiction, hence $p \notin P_I$.
To see that p is an accumulation point, consider that for any $n \in \mathbb{N}$, the point $q_n := (n, \lfloor \sqrt{2}n \rfloor) \in P_I$, and so $p_n := \frac{1}{n} q_n + (1 - \frac{1}{n})0 \in P_I$. Furthermore,

$$\|p_n - p\| = |\sqrt{2} - \frac{\lfloor \sqrt{2}n \rfloor}{n}| = \frac{1}{n} |\sqrt{2}n - \lfloor \sqrt{2}n \rfloor| \to 0$$

(b) For $P = \text{conv}\{(0,0), (0,1), (k, \frac{1}{2})\}$ we get $P^{(i)} \neq P_I$ for all $i < 2k$ because in each iteration the width of $P^{(i)}$ is reduced by only $\frac{1}{2}$.

The answers to this assignment sheet will not be marked by the tutors. Sketches of the solutions will be published on the web page of the exercises on July 11 after the lecture.