Linear and Integer Optimization
Assignment Sheet 13

1. Show that $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not totally unimodular but $\{x \in \mathbb{R}^3 \mid Ax = b\}$ is integral for all integral vectors b.

2. Let $A \in \{0,1\}^{m \times n}$ be a matrix where in each column the 1's are arranged consecutively, i.e. for each column $j \in \{1, \dots, n\}$ there are $i_1^j, i_2^j \in \{1, \dots, m\}$ s.t.:
$$a_{ij} = \begin{cases} 1, & i_1^j \le i \le i_2^j \\ 0, & \text{else} \end{cases}$$
for $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, m\}$ (if $i_1^j > i_2^j$, the column consists of zeros only). Show that A is totally unimodular.

3. Consider the following problem: We are given a directed graph G and nodes $s, t \in V(G)$ with $s \neq t$. Moreover, we are given integral mappings $l, u : E(G) \to \mathbb{Z}$ such that $l(e) \le u(e)$ for all $e \in E(G)$. The task is to find a mapping $f : E(G) \to \mathbb{R}$ with $l(e) \le f(e) \le u(e)$ for all edges $e \in E(G)$ and $\sum_{e \in \delta_G^-(v)} f(e) = \sum_{e \in \delta_G^+(v)} f(e)$ for all $v \in V(G) \setminus \{s, t\}$ such that $\sum_{e \in \delta_G^+(s)} f(e) - \sum_{e \in \delta_G^-(s)} f(e)$ is maximized. This problem generalizes the max-flow problem. Show that there is always an integral optimum solution and show that the value of a maximum solution equals
$$\min \left\{ \sum_{e \in \delta_G^+(X)} u(e) - \sum_{e \in \delta_G^-(X)} l(e) \mid X \subseteq V(G) \setminus \{t\}, s \in X \right\}.$$

4. (a) Give an example of a polyhedron with $P_I \neq P^{(i)}$ for all $i \in \mathbb{N}$.
   (b) Show that for any $k \in \mathbb{N}$ there is a rational polyhedron such that $P_I \neq P^{(i)}$ for all $i \in \{1, \dots, k\}$.

The answers to this assignment sheet will not be marked by the tutors. Sketches of the solutions will be published on the web page of the exercises on July 11 after the lecture.