Summer term 2019 Research Institute for Discrete Mathematics
Dr. U. Brenner University of Bonn

Linear and Integer Optimization
Assignment Sheet 12

1. Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron with $P = P_I$. Show that for any rational vector c there is a polynomial-time algorithm computing a vector $x \in P \cap \mathbb{Z}^n$ maximizing $c^t x$ over $P \cap \mathbb{Z}^n$, provided that there is an optimum solution. (4 points)

2. (a) Show that the systems
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
and
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
define the same polyhedron. Prove that the first system is TDI while the second one is not TDI.

   (b) Prove or disprove the following statement: If $Ax \leq b$ (with $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$) is TDI and $\alpha \in \mathbb{Q}_{>0}$, then $\alpha A x \leq \alpha b$ is TDI. (2+2 points)

3. Let $a = (a_1, \dots, a_n) \in (\mathbb{N} \setminus \{0\})^n$ be a vector and β a rational number. Prove that $a^t x \leq \beta$ is TDI if and only if the greatest common divisor of $a_1, \dots, a_n$ is 1. (2 points)

4. Prove the following statement: If $Ax \leq b$ is a TDI-system, and $a^t x \leq \beta$ is valid for any $x \in \mathbb{R}^n$ with $Ax \leq b$, then the system $Ax \leq b, a^t x \leq \beta$ is also totally dual integral. (4 points)

5. (a) Show that a non-empty polyhedral cone C is pointed if and only if there is a vector b such that $b^t x > 0$ for all $x \in C \setminus \{0\}$.

   (b) Let C b a non-empty rational pointed polyhedral cone. Show that there is a unique minimum integral Hilbert basis generating C. (3+3 points)

   Hint: For (b) consider the integral vectors in C that cannot be written as the sum of two other integral vectors in C.

Due date: Thursday, July 4, 2019, before the lecture.

The results of this assignment sheet do not count for the admission to the final exam.