Summer term 2019 — Research Institute for Discrete Mathematics
Dr. U. Brenner — University of Bonn

Linear and Integer Optimization
Assignment Sheet 10

1. Show that $|\det(A)| \leq \prod_{i=1}^{n} \|a_i\|$ for an $n \times n$-matrix A with columns $a_1, \ldots, a_n$ (where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is the standard Euclidean norm). (3 points)

2. Let $K \subseteq \mathbb{R}^n$ be an r-R-sandwiched convex set, $c \in \mathbb{R}^n$, $\delta = \sup\{c^t x \mid x \in K\}$, and $0 < \epsilon < \delta$. Moreover, let $U = \{x \in K \mid c^t x \geq \delta - \epsilon\}$. Prove that
$$\text{volume}\,(U) \geq \left(\frac{\epsilon}{2\|c\|R|}\right)^{n-1} r^{n-1} \frac{1}{n^n} \frac{\epsilon}{2\|c\|} \frac{1}{n}.$$
(5 points)

3. Assume that you are given an oracle with polynomial running time that computes an optimum solution in K for any linear objective function. Show that there is a separation oracle with polynomial running time for $K^* := \{y \in \mathbb{R}^n \mid y^t x \leq 1 \text{ for all } x \in K\}$. (3 points)

4. Consider the following situation: You are a foreign exchange trader trading in n different currencies with an initial budget $b = (b_1, \ldots, b_n) \geq 0$, where b_i is the amount of currency i. For every pair $1 \leq i, j \leq n$ of currencies, you are given an exchange rate r_{ij}. Assume that you are able to exchange an arbitrary (non-negative) amount x_{ij} of currency i into $r_{ij} x_{ij}$ units of currency j. You want to determine trades that will maximize the amount of currency 1 available to you in the end. You may assume that all trades can occur simultaneously (e.g. by cost-free borrowing for the instant during which the trades occur), so long as the final amounts in all currencies are non-negative.

   Model this problem as a linear program and derive its dual. (4 points)

   Remark: If the LP is unbounded, one speaks of an *arbitrage opportunity*, i.e. a situation that allows "risk-free" profit.

Due date: **Tuesday**, June 18, 2019, before the lecture.