Summer term 2019 | Research Institute for Discrete Mathematics
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# Linear and Integer Optimization
## Assignment Sheet 9

1. Let $A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & -1 \end{pmatrix}$ and $b := \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.
   Use the IDEALIZED ELLIPSOID ALGORITHM with $R = 2$ to compute a feasible solution in $P = \{x \in \mathbb{R}^2 \mid Ax \leq b\}$ for $s = -1$ and for $s = -2$. (4 points)

2. A semidefinite program is an optimization problem

$$\begin{aligned} \min\ & C \star X \\ & A_i \star X \leq b_i \qquad \forall i = 1, \ldots, m \\ & X \succeq 0 \\ & X \in \mathbb{R}^{n \times n} \end{aligned}$$

   where $C, A_1, \ldots, A_n$ are matrices, $A \star X := \sum_{1 \leq i,j \leq n} a_{ij} x_{ij}$ and $X \succeq 0$ means that $X$ is symmetric and positiv semidefinite.

   (a) Show that the set $\{X \in \mathbb{R}^{n \times n} \mid X \succeq 0\}$ is a closed cone.

   (b) Construct a polynomial-time separation oracle for this set. (You may assume that you can compute basic arithmetic operations on real numbers, including square roots, exactly and in constant time.) (3+4 points)

3. Consider the following optimization problem:

$$\begin{aligned} \min\ & \frac{(c^t x)^2}{d^t x} \\ \text{s.t.}\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

   where $c, d \in \mathbb{R}^n$, $A \in R^{m \times n}$ and $b \in \mathbb{R}^m$ are given such that $d^t x > 0$ for any $x \in \mathbb{R}^n$ with $Ax \geq b$ and $x \geq 0$. Show that this problem can be written as a semidefinite program (see the previous exercise). (5 points)
   Hint: Replace the objective function by "$k$" (where $k$ is a new variable) and add the constraint $k \geq \frac{(c^t x)^2}{d^t x}$.

4. Let $G$ be simple undirected graph. Consider the following linear program:

$$\begin{array}{llll} \min & \sum_{e=\{v,w\}\in E(G)} x_{vw} & & \\ \text{s.t.} & \sum_{w\in S} x_{vw} \geq & \lceil \frac{1}{4}|S|^2 + \frac{1}{2}|S| \rceil & \text{for } v \in V(G), S \subseteq V(G) \setminus \{v\} \\ & x_{uw} \leq & x_{uv} + x_{vw} & \text{for } u, v, w \in V(G) \\ & x_{vw} \geq & 0 & \text{for } v \in V(G) \\ & x_{vv} = & 0 & \text{for } v \in V(G) \end{array}$$

(a) Show that this is a relaxation of the following problem: Find distances $x_{vw}$ for the nodes of $G$ such that $\sum_{e=\{v,w\}\in E(G)} x_{vw}$ is minimized under the condition that there is an ordering $\{v_1, \ldots, v_{|V(G)|}\} = V(G)$ with $x_{v_i v_j} = |i - j|$ for $i, j \in \{1, \ldots, |V(G)|\}$.

(b) Prove that there is a separation oracle for the polyhedron of the feasible solutions of the LP with running polynomial in the number of nodes and edges of $G$. (2+2 points)

Due date: Thursday, June 6, 2019, before the lecture.