Summer term 2019 Research Institute for Discrete Mathematics
Dr. U. Brenner University of Bonn

Linear and Integer Optimization
Assignment Sheet 7

1. Let $H = (V, E)$ be a hypergraph, so V is a finite set of nodes and $E \subseteq 2^V$. Assume that you are given $F \subseteq V$ and $x, y : F \to \mathbb{R}$.

   (a) Describe the following problem as a linear program. We ask for an extension $x, y : V \setminus F \to \mathbb{R}$ such that
   $$\sum_{e \in E} \left(\max_{v \in e} x(v) - \min_{v \in e} x(v) + \max_{v \in e} y(v) - \min_{v \in e} y(v) \right)$$
   is minimized.

   (b) Dualize the LP from (a) and show that the dual LP is equivalent to a Min-Cost Flow Problem. (2+3 points)

   Remark: This is a relaxation of the placement problem in chip design. The vertices correspond to connected modules that must be placed minimizing the length of all interconnects (hyperedges). Vertices in F are preplaced. The problem becomes much harder when requiring disjointness of the modules.

2. Describe an algorithm for the following problem: Given a tree T, you have time $O(|V(T)|)$ for some preprocessing. After the preprocessing, you should be able to compute for any two given nodes x and y of T in time $O(\mathrm{dist}_T(x, y))$ the x-y-path in T. (5 points)
   Remark: This is a problem that has to be solved during the Network Simplex Algorithm when computing a fundamental circuit.

3. Consider a linear program $\max\{c^t x \mid Ax = b, x \geq 0\}$ such that $A \in \mathbb{R}^{m \times n}$, $\mathrm{rank}(A) = m$ and $Ax = b$ is feasible. Let B be a dual feasible basis, i.e. a basis such that $\tilde{y} = (A_B^t)^{-1} c_B$ is a feasible solution of the dual LP.

   (a) Show that the entry z_0 of the simplex tableau $T(B)$ is the cost of the dual solution.

   (b) Let $\beta \in B$ with $p_\beta < 0$ and $\alpha \in N$ with $q_{\beta\alpha} > 0$ such that $\frac{r_\alpha}{q_{\beta\alpha}} \geq \frac{r_j}{q_{\beta j}}$ for all $j \in N$ with $q_{\beta j} > 0$. Prove that $(B \setminus \{\beta\}) \cup \{\alpha\}$ is a dual feasible basis. Moreover, show that the value of the dual solution is changed by $\frac{-p_\beta}{q_{\beta\alpha}} r_\alpha$. (1+3 points)

4. Let (G, u, b, c) be an instance of the Minimum-Cost Flow Problem.

   (a) Dualize the linear program formulation of the Minimum-Cost Flow Problem that was presented in the lecture.

   (b) Let (r, T, L, U) be a feasible spanning tree structure for (G, u, b, c), and let f be the flow and π the potential associated to it. Show by considering the complementary slackness constraints that f is optimum if $c_\pi(e) \geq 0$ for all $e \in L$ and $c_\pi(e) \leq 0$ for all $e \in U$. (3+3 points)

Due date: Thursday, May 23, 2019, before the lecture.