Linear and Integer Optimization
Assignment Sheet 6

1. Consider the following linear program:

$$\begin{array}{lrcrcrcrcrcrcl} \max & 9x_1 & + & 3x_2 & + & x_3 & & & & & & & & \\ \text{s.d.} & x_1 & & & & & + & x_4 & & & & & = & 1 \\ & 6x_1 & + & x_2 & & & & & + & x_5 & & & = & 9 \\ & 18x_1 & + & 6x_2 & + & x_3 & & & & & + & x_6 & = & 81 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0 \end{array}$$

Show that the SIMPLEX ALGORITHM using the "largest coefficient rule" as pivot rule and starting with the basis $\{4, 5, 6\}$ considers all vertices of the solution polyhedron before finding an optimum solution. (4 points)

2. Consider a linear program $\max\{c^t x \mid Ax = b, x \geq 0\}$ such that $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$ and $Ax = b$ is feasible. Prove or disprove the following statements about the SIMLEX ALGORITHM:
   (a) A variable that has just entered the basis can leave the basis in the next iteration.
   (b) A variable that has just left the basis can enter the basis in the next iteration.
   (c) If x is unique optimum basic solution and $\tilde{x}$ a second best basic solution with strictly smaller solution value then x can be computed from $\tilde{x}$ by exchanging one basic variable.
   (d) If no basic solution is degenerated and the LP is feasible and bounded then there is a unique optimum solution. (1+1+1+1 points)

3. Consider a linear program $\max\{c^t x \mid Ax = b, x \geq 0\}$ such that $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$ and $Ax = b$ is feasible. Let B be a feasible basis with basic solution x^* and reduced cost vector $r \leq 0$ (so x^* is an optimum solution). Let $I = \{j \in N \mid r_j = 0\}$.
   (a) Prove that x^* is the unique optimum solution if $I = \emptyset$.
   (b) Assume that $I \neq \emptyset$. Prove that in this case x^* is the unique optimum solution if and only if the following linear program has the optimum solution value 0:

$$\begin{array}{llcll} \max & \sum_{i \in I} x_i & & & \\ \text{s.t.} & Ax & = & b & \\ & x_i & = & 0 & \text{for } i \in N \setminus I \\ & x_i & \geq & 0 & \text{for } i \in B \cup I \end{array}$$

(4 points)

4. For $n \in \mathbb{N} \setminus \{0\}$ and a subset $X \subseteq \mathbb{R}$ let

$$M_X = \left\{ A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \mid a_{i_0 j_0} \in X, \sum_{i=1}^{n} a_{ij_0} = 1, \sum_{j=1}^{n} a_{i_0 j} = 1 \quad (\text{for } i_0, j_0 \in \{1, \dots, n\}) \right\}.$$

Show that an $n \times n$-matrix A is in $M_{\mathbb{R}_{\geq 0}}$ if and only if it is a convex combination of matrices in $M_{\{0,1\}}$. (4 points)

5. Consider the linear program (P)

$$\begin{array}{rrcl} & \max c^t x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

Let $\tilde{x}$ be a solution of (P) such that there is a subsystem $A'x \leq b'$ with $A'\tilde{x} = b'$ where A' consists of n linearly independent rows of A. Moreover, assume that for all constraints $a^t x \leq \beta$ in $Ax \leq b$ but not in $A'x \leq b'$ we have $a^t \tilde{x} < \beta$. Let $\delta = c^t \tilde{x}$. In addition, let $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_m)$ be an optimum solution of the dual LP. Show that there is a positive ϵ such that for any vector $p = (p_1, \dots, p_m)$ with $p_i \in [0, \epsilon]$ $(i = 1, \dots, m)$ the modified linear program (P')

$$\begin{array}{rrcl} & \max c^t x & & \\ \text{s.t.} & Ax & \leq & b + p \\ & x & \geq & 0 \end{array}$$

has an optimum solution of value $\delta + \tilde{y}^t p$. (4 points)

Due date: Thursday, May 16, 2019, before the lecture.